1/12

Math 155, *Lecture Notes- Bonds* — Name____________

Section 8.8 Improper Integrals

When we have previously considered the definite integral $\int_a^b f(x)\,dx$, we know that we are looking at a bounded area, and that we will need to use the Fundamental Theorem of Calculus if we are to apply a calculus technique to evaluate the integral. In particular, the Fundamental Theorem of Calculus will require that the interval $[a,b]$ is finite and the integrand, $f(x)$, is continuous on the interval $[a,b]$.

In this section we will study integrals that have issues with the Fundamental Theorem of Calculus. That is, we will consider **improper integrals**, or the areas of unbounded regions between the integrand, $f(x)$, and the x-axis. Two main obstacles will need to be overcome. First, we will consider improper integrals with infinite integration limits. Second, we will consider improper integrals with infinite discontinuities.

Definition of Improper Integrals with Infinite Integration Limits

1. If f is continuous on the interval $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. If f is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. If f is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

where c is any real number (see Exercise 110).

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

When one or both of the limits of integration are infinity, are we working with infinite integration limits, and we are considering an integrand over an infinite interval. In general, we'll deal with these types of integrals by replacing the infinity symbol with a variable (usually b), antidifferentiating the integrand, and then we'll take the limit of the result as b goes to infinity.

Ex.1 Evaluate: $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$

use the definition with the limit.

$$= \lim_{b\to\infty} \int_1^{b} x^{-2}\,dx$$

$$= \lim_{b\to\infty} \left[-1\cdot x^{-1}\right]_1^b$$

$$= -\lim_{b\to\infty} \left[\frac{1}{x}\right]_1^b$$

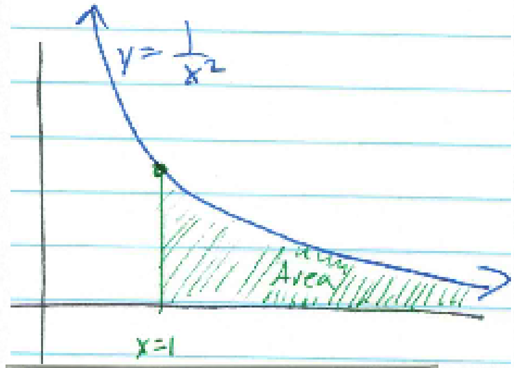
$$= -\lim_{b\to\infty} \left[\frac{1}{b} - 1\right]$$

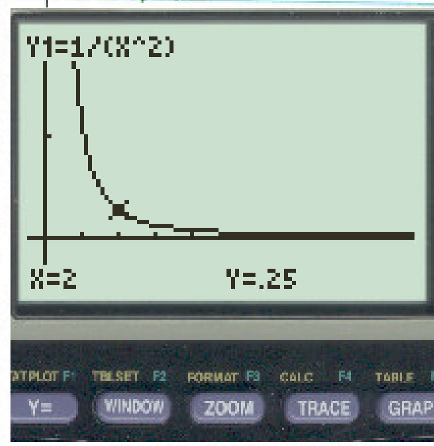
$$= -\left[0-1\right]$$

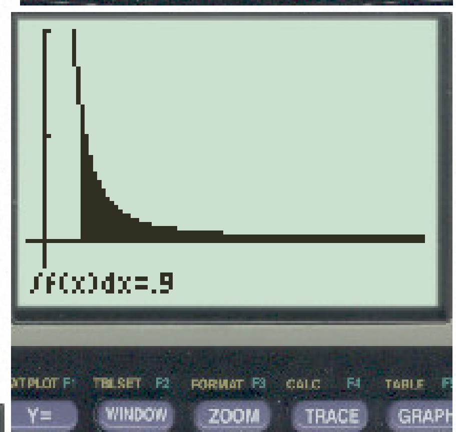
$$= 1$$

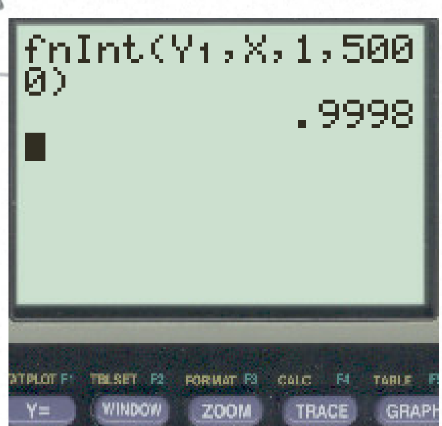
$$\boxed{\int_1^{\infty} \frac{1}{x^2}\,dx = 1}$$

The area bounded by the curve, $x=1$, and the x-axis is finite.





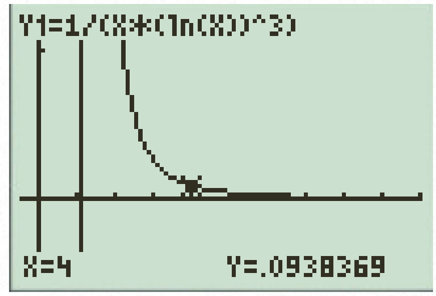




It is interesting to notice that the area under a curve on an infinite interval was not infinity, as we might have assumed it to be. The area value was a small number. This won't always be the case, but it is important enough to point out that not all areas on an infinite interval will yield infinite areas.

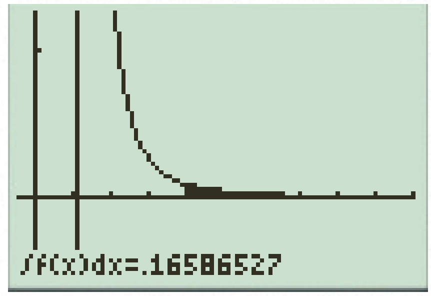
Ex.2 Evaluate: $\displaystyle\int_4^{\infty} \frac{1}{x[\ln(x)]^3}\,dx$

use the definition with the limit.



$$= \lim_{b\to\infty} \int_4^{b} \frac{1}{x[\ln(x)]^3}\,dx$$

$$= \lim_{b\to\infty} \int_{u=\ln(4)}^{u=\ln(b)} \frac{1}{x\cdot u^3}\cdot(x\cdot du)$$



Let $u=\ln(x)$

$\frac{du}{dx}=\frac{1}{x}$

$x\,du=dx$

if $x=b$
$u=\ln(b)$

if $x=4$
$u=\ln(4)$

$\ln(4)=\ln(2^2)$
$\ln(4)=2\ln(2)$

$$= \lim_{b\to\infty} \int_{\ln(4)}^{\ln(b)} u^{-3}\,du$$

$$= \lim_{b\to\infty} \left[\frac{u^{-2}}{-2}\right]_{\ln(4)}^{\ln(b)}$$

$$= -\frac{1}{2}\cdot \lim_{b\to\infty} \left[\frac{1}{u^2}\right]_{\ln(4)}^{\ln(b)}$$

$$= -\frac{1}{2}\cdot \lim_{b\to\infty} \left[\frac{1}{[\ln(b)]^2} - \frac{1}{[\ln(4)]^2}\right]$$

$$= -\frac{1}{2}\cdot\left[0 - \frac{1}{[2\ln(2)]^2}\right]$$

```
fnInt(Y1,X,4,25000)
        .2552953884
```

$$= -\frac{1}{2}\left[-\frac{1}{2^2\ln^2(2)}\right]$$

$$= \frac{1}{8\ln^2(2)}$$

```
1/(8(ln(2))^2)
        .2601711226
```

$$\boxed{\int_4^{\infty} \frac{1}{x[\ln(x)]^3}\,dx = \frac{1}{8\ln^2(2)}}$$

The area bounded by the curve, $x=4$, and the x-axis is finite.

Ex.3 Evaluate: $\int_0^\infty xe^{-\frac{x}{2}}dx$

Use the definition with the limit.

$$= \lim_{b\to\infty} \int_0^b xe^{-\frac{x}{2}}dx$$

$$= \lim_{b\to\infty} \left[\overset{u}{(x)}\overset{v}{\left(-2e^{-\frac{x}{2}}\right)}\Big|_0^b - \int_0^b \overset{v}{\left(-2e^{-\frac{x}{2}}\right)}\cdot\overset{du}{(dx)} \right]$$

$$= \lim_{b\to\infty} \left\{ \left[-2xe^{-\frac{x}{2}}\right]_0^b + 2\int_0^b e^{-\frac{x}{2}}dx \right\}$$

$$= \lim_{b\to\infty} \left\{ \left[-2xe^{-\frac{x}{2}}\right]_0^b + 2\left[-2e^{-\frac{x}{2}}\right]_0^b \right\}$$

$$= \lim_{b\to\infty} \left\{ \left[-2xe^{-\frac{x}{2}} - 4e^{-\frac{x}{2}}\right]_0^b \right\}$$

$$= -2\cdot\lim_{b\to\infty} \left\{ \left[e^{-\frac{x}{2}}(x+2)\right]_0^b \right\}$$

$$= -2\cdot\lim_{b\to\infty} \left\{ e^{-\frac{b}{2}}(b+2) - e^{-\frac{0}{2}}(0+2) \right\}$$

$$= -2\lim_{b\to\infty} \left\{ \frac{b+2}{e^{\frac{b}{2}}} - 1\cdot 2 \right\}$$

$$= -2\boxed{\lim_{b\to\infty} \frac{b+2}{e^{\frac{b}{2}}}} + 2\lim_{b\to\infty} 2 \quad ★★$$

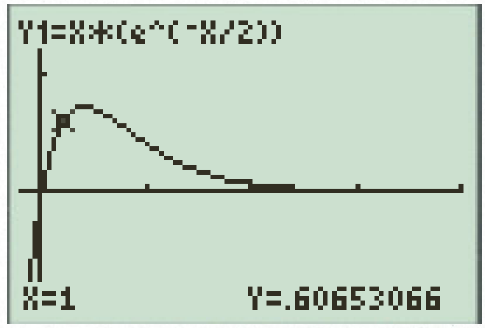
$$= -2\lim_{b\to\infty} \frac{\frac{d}{db}(b+2)}{\frac{d}{db}\left(e^{b/2}\right)} + 2\cdot 2$$

Using L'Hôpital's Rule

$$= 2\lim_{b\to\infty} \frac{1}{\frac{1}{2}e^{b/2}} + 4$$

$$= 0+4$$

$$= \underline{4} \checkmark$$



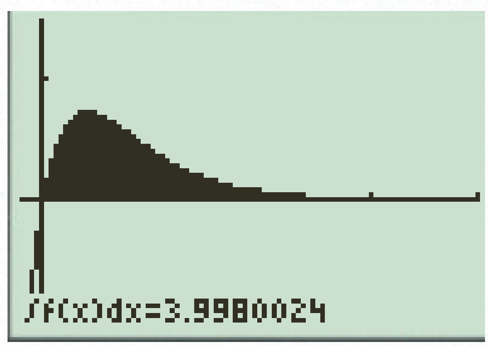
$$\boxed{\int u\,dv = uv - \int v\,du}$$

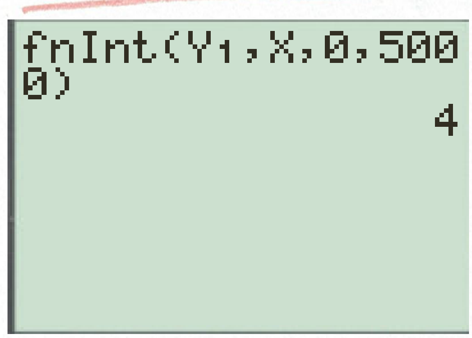
Integration by Parts:

Let $u = x$	$dv = e^{-\frac{x}{2}}dx$
$\frac{du}{dx} = 1$	$\int dv = \int e^{-\frac{x}{2}}dx$
$du = dx$	$v = -2e^{-\frac{x}{2}} + C$
	$v = -2e^{-\frac{x}{2}}$

★★ $\lim_{b\to\infty} \frac{b+2}{e^{b/2}} = \boxed{\frac{\infty}{\infty}}$ Stop!

Indeterminate Form





The second type of improper integral that we'll be looking at is one that has an infinite discontinuity ***<u>at or between</u>*** the limits of integration. Essentially, these are integrals that have discontinuous integrands and we have difficulty applying the Fundamental Theorem of Calculus to these integrals. The process we will use to work with these integrals is here is basically the same, but with one subtle difference.

Definition of Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

2. If f is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

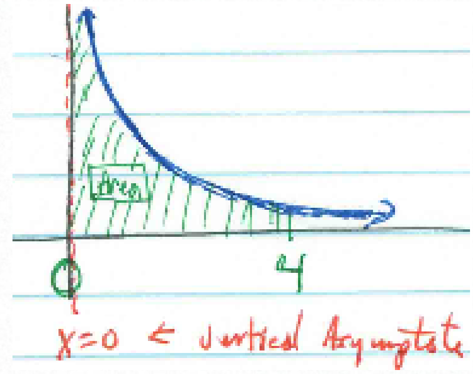
3. If f is continuous on the interval $[a, b]$, except for some c in (a, b) at which f has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Please note that the limits in these cases are required to make use of **<u>right or left handed limits</u>**. This is the way are assured that we are indeed working inside the interval defined by the limits of integration. This is the reason we are using one-sided limits, which need to be signified by using +, or – in the right superscript. In general, we'll deal with these types of integrals by replacing the *b numeral* (where the discontinuity exists) with a variable (usually *c*), antidifferentiating the integrand, and then we'll take the limit of the result as *c* approaches *b* from inside the interval.

Ex.4 Evaluate: $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$



$$= \lim_{c \to 0^+} \int_c^4 \frac{1}{\sqrt{x}}\,dx$$

use the definition with the limit.

$$= \lim_{c \to 0^+} \int_c^4 x^{-\frac{1}{2}}\,dx$$

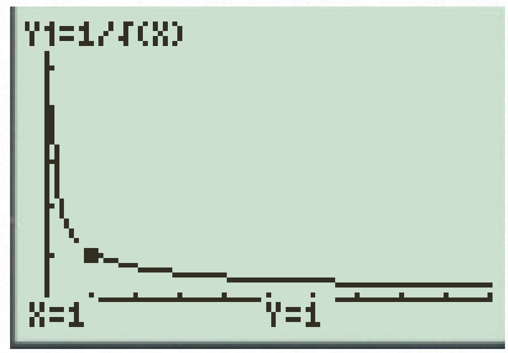
$$= \lim_{c \to 0^+} \left[\frac{2}{1} \cdot x^{\frac{1}{2}}\right]_c^4$$

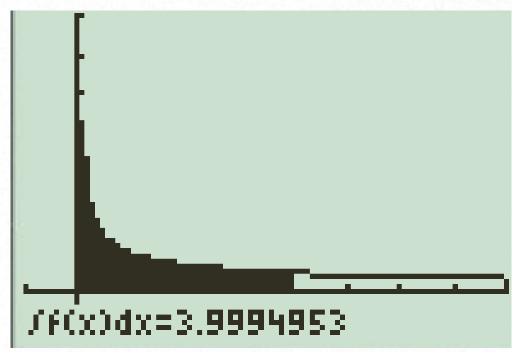
$$= 2 \cdot \lim_{c \to 0^+} \left[\sqrt{4} - \sqrt{c}\right]$$

$$= 2 \cdot (2 - 0)$$

$$= 4$$

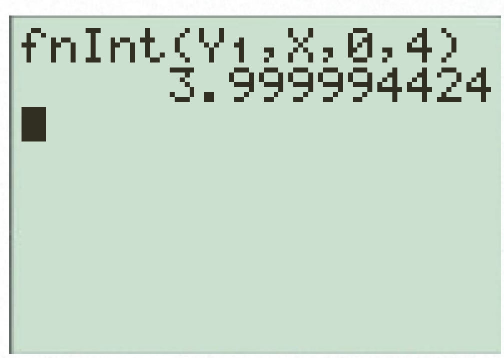
$$\boxed{\int_0^4 \frac{1}{\sqrt{x}}\,dx = 4}$$





The area bounded by the curve, $x=0$, $x=4$, and the x-axis is finite.

We can say the integral converges because the limit exists.



Ex.5 Evaluate: $\displaystyle\int_3^4 \frac{1}{(x-3)^{\frac{3}{2}}}\,dx$

$$= \lim_{c\to 3^+} \int_c^4 (x-3)^{-\frac{3}{2}}\,dx$$

use the definition with the limit.

$$= \lim_{c\to 3^+} \int_{u=c-3}^{u=1} u^{-\frac{3}{2}}\,du$$

$$= \lim_{c\to 3^+} \left[-\frac{2}{1}\cdot u^{-\frac{1}{2}} \right]_{c-3}^{1}$$

$$= \lim_{c\to 3^+} \left[\frac{-2}{\sqrt{u}} \right]_{c-3}^{1}$$

$$= \lim_{c\to 3^+} \left[\frac{-2}{\sqrt{1}} + \frac{2}{\sqrt{c-3}} \right]$$

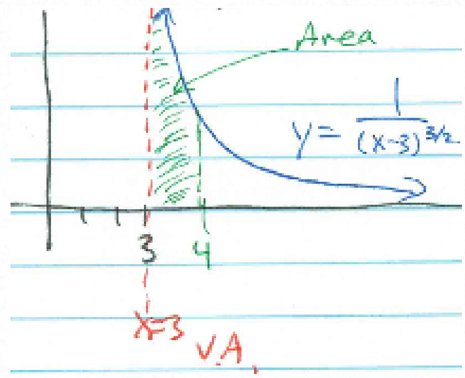
$$= -2 + \lim_{c\to 3^+} \frac{2}{\sqrt{c-3}}$$

$$= -2 + \infty$$

$$= \infty$$

$$\boxed{\int_3^4 \frac{1}{(x-3)^{\frac{3}{2}}}\,dx = \infty}$$

Since the area is infinite, we can say the integral diverges.

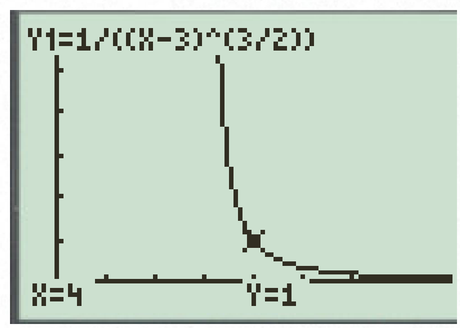


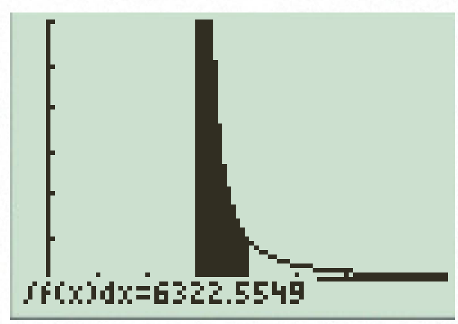
Let $u = x-3$

$\frac{du}{dx} = 1$

$du = dx$

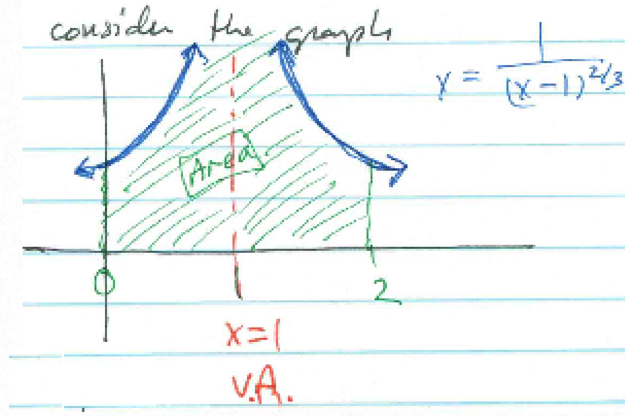
if $x = 4$	if $x = c$
$u = (4)-3$	$u = c-3$
$u = 1$	






```
fnInt(Y1,X,3.000
0001,4)
          6322.554978
```

Ex.6 Evaluate: $\displaystyle\int_0^2 \frac{1}{(x-1)^{\frac{2}{3}}}dx$

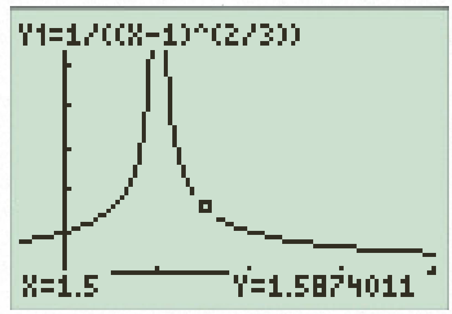


$= \displaystyle\int_0^1 (x-1)^{-2/3}dx + \int_1^2 (x-1)^{-2/3}dx$

$= \displaystyle\lim_{b\to 1^-}\int_0^b (x-1)^{-2/3}dx + \lim_{c\to 1^+}\int_c^2 (x-1)^{-2/3}dx$

use the definition with the limit.

$= \displaystyle\lim_{b\to 1^-}\int_{u=-1}^{u=b-1} u^{-2/3}du + \lim_{c\to 1^+}\int_{u=c-1}^{u=1} u^{-2/3}du$



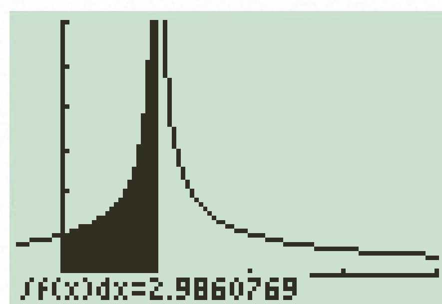
$= \displaystyle\lim_{b\to 1^-}\left[\frac{3}{1}u^{\frac{1}{3}}\right]_{-1}^{b-1} + \lim_{c\to 1^+}\left[3u^{1/3}\right]_{c-1}^{1}$

$= 3\cdot\displaystyle\lim_{b\to 1^-}\left[\sqrt[3]{b-1} - \sqrt[3]{-1}\right] + 3\cdot\lim_{c\to 1^+}\left[\sqrt[3]{1} - \sqrt[3]{c-1}\right]$

$= 3\cdot[0+1] + 3\cdot[1-0]$

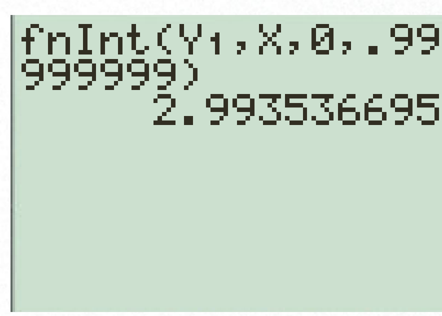
$= 3+3$

$= 6$



Since the area is finite, we can say the integral converges.

$\boxed{\displaystyle\int_0^2 \frac{1}{(x-1)^{2/3}}dx = 6}$



Ex.7 Evaluate: $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$

Use the definition with the limit.

$$= \lim_{c\to 2^-}\int_0^c \frac{1}{\sqrt{2^2-x^2}}\,dx$$

arcsine? ★

$$= \lim_{c\to 2^-}\left[\arcsin\left(\frac{x}{2}\right)\right]_0^c$$

$$= \lim_{c\to 2^-}\left[\arcsin\left(\frac{c}{2}\right)-\arcsin\left(\frac{0}{2}\right)\right]$$

$$= \arcsin\left(\frac{2}{2}\right)-\arcsin(0)$$

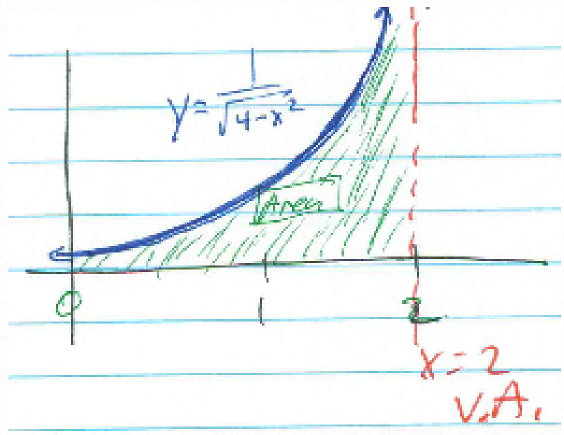
$$= \arcsin(1) - 0$$

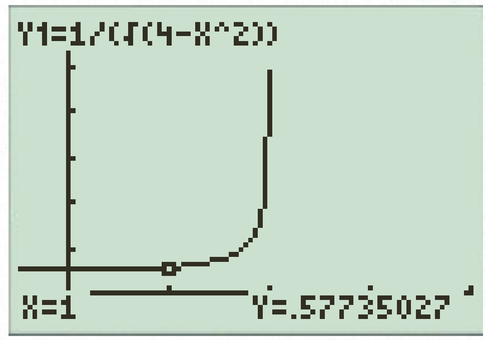
$$= \frac{\pi}{2} - 0$$

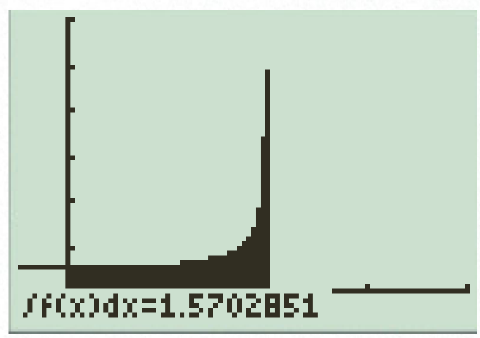
$$= \frac{\pi}{2}$$

$$\boxed{\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \frac{\pi}{2}}$$

Since the area is finite, we can say the integral converges.








```
fnInt(Y1,X,0,1.9
9999999)
      1.570696338
π/2
      1.570796327
```

★ $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du$

$= \arcsin\left(\frac{u}{a}\right) + C$

Let $\theta = \arcsin(0)$

$\sin(\theta) = 0$

$\theta = 0$ ✓

Let $\alpha = \arcsin(1)$

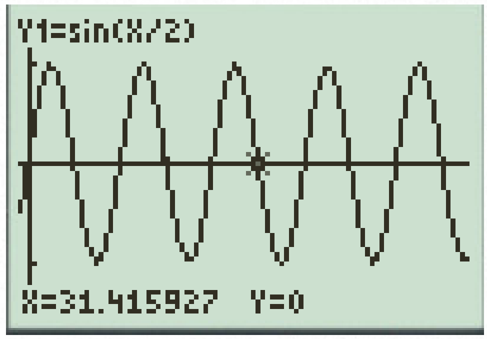
$\sin(\alpha) = 1$

$\alpha = \frac{\pi}{2}$

Ex.8 Evaluate: $\displaystyle\int_0^{\infty} \sin\left(\frac{x}{2}\right)dx$

$$= \lim_{b\to\infty} \int_0^{b} \sin\left(\frac{x}{2}\right)dx$$

use the definition with the limit.



Let $u = \frac{x}{2}$

$\frac{du}{dx} = \frac{1}{2}$

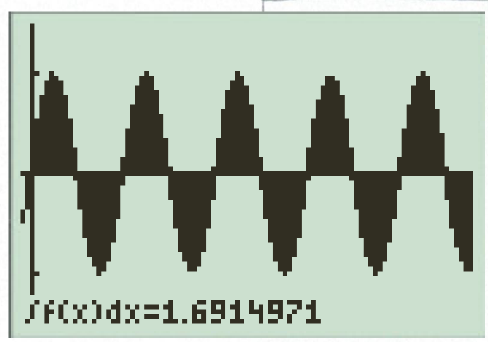
$2du = dx$

if $x=b$	if $x=0$
$u=\frac{b}{2}$	$u=\frac{0}{2}$
	$u=0$

$$= \lim_{b\to\infty} \int_{u=0}^{u=\frac{b}{2}} \sin(u)(2du)$$

$$= 2\cdot\lim_{b\to\infty} \int_0^{\frac{b}{2}} \sin(u)\,du$$

$$= 2\lim_{b\to\infty} \Big[-\cos(u)\Big]_0^{\frac{b}{2}}$$

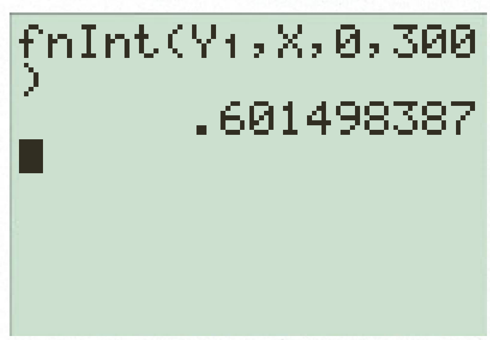


$$= -2\lim_{b\to\infty} \left[\cos\left(\frac{b}{2}\right) - \cos(0)\right]$$

$$= -2\lim_{b\to\infty} \left[\cos\left(\frac{b}{2}\right) - 1\right]$$

$$= \text{"D.N.E."}$$

This limit does not exist because the cosine function oscillates between -1 & 1 as b goes toward ∞.



The improper integral diverges and the corresponding area is infinite.

In most examples in a Calculus II class that are worked over infinite intervals the limit either exists, or is infinite. However, there are limits that don't exist, as the previous example showed, so don't forget about those.

Ex.9 Evaluate: $\displaystyle\int_0^{\infty} \frac{e^x}{1+e^x}dx$

use the definition with the limit.

$$= \lim_{b\to\infty} \int_0^b \frac{e^x}{1+e^x}dx$$

$$= \lim_{b\to\infty} \int_{u=2}^{u=1+e^b} \frac{e^x}{u}\left(\frac{du}{e^x}\right)$$

$$= \lim_{b\to\infty} \int_2^{1+e^b} \frac{1}{u}du$$

$$= \lim_{b\to\infty} \Big[\ln|u|\Big]_2^{1+e^b}$$

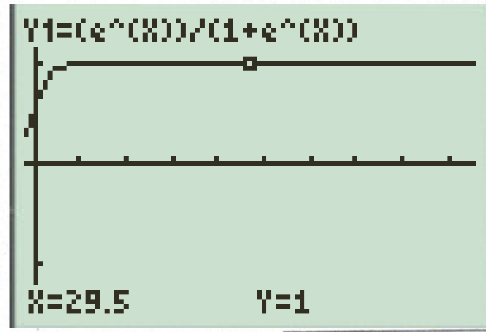
$$= \lim_{b\to\infty} \left[\ln(1+e^b) - \ln(2)\right]$$

$$= \infty - \ln(2)$$

$$= \infty \checkmark$$

$$\boxed{\int_0^{\infty} \frac{e^x}{1+e^x}dx = \infty}$$

This limit does not exist, the improper integral diverges, and the corresponding area is infinite.

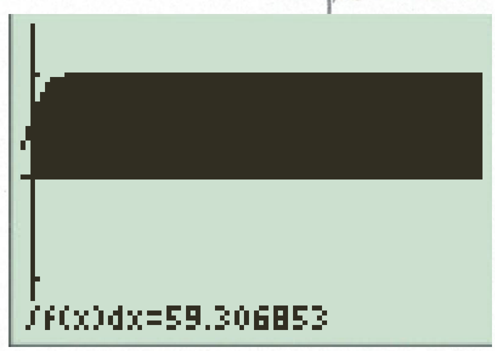


Let $u = 1+e^x$

$\dfrac{du}{dx} = e^x$

$\dfrac{du}{e^x} = dx$

if $x=0$	if $x=b$
$u=1+e^{(0)}$	$u=1+e^b$
$u=1+1$	
$u=2$	




```
fnInt(Y1,X,0,100
)
          99.30685282
```

An interesting phenomenon to notice is when thinking in terms of area defined by $\int_1^\infty \frac{1}{x}dx$ we can use antidifferentiation and limits to find that the area is infinite. On the other hand, the area defined by $\int_1^\infty \frac{1}{x^2}dx$ was quite small. There really isn't all that much difference between these two integrands and yet there is a large difference in the area under them. We can actually extend this out to the following theorem.

THEOREM 8.5 A Special Type of Improper Integral

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1}, & \text{if } p > 1 \\ \text{diverges}, & \text{if } p \le 1 \end{cases}$$

Remember, we will call improper integrals **convergent** if the associated limit exists and is a finite number (*i.e.* it's not plus or minus infinity), and **divergent** if the associated limit either doesn't exist or is (plus or minus) infinity.